

2000]Primary 55P35, 17B35; Secondary 55Q05

A TORSION-FREE MILNOR-MOORE THEOREM

JONATHAN A. SCOTT

ABSTRACT. Let ΩX be the space of Moore loops on a finite, q -connected, n -dimensional CW complex X , and let $R \subset \mathbf{Q}$ be a subring containing $1/2$. Let $\rho(R)$ be the least non-invertible prime in R . For a graded R -module M of finite type, let $FM = M/\text{Torsion } M$. We show that the inclusion $P \subset FH_*(\Omega X; R)$ of the sub-Lie algebra of primitive elements induces an isomorphism of Hopf algebras $UP \xrightarrow{\cong} FH_*(\Omega X; R)$, provided $\rho(R) \geq n/q$. Furthermore, the Hurewicz homomorphism induces an embedding of $F(\pi_*(\Omega X) \otimes R)$ in P , with $P/F(\pi_*(\Omega X) \otimes R)$ torsion. As a corollary, if X is elliptic, then $FH_*(\Omega X; R)$ is a finitely generated R -algebra.

1. INTRODUCTION

Let ΩX be the Moore loop space on a pointed, simply-connected topological space X . A theorem of Milnor and Moore [14] states that $H_*(\Omega X; \mathbf{Q})$ is the universal enveloping algebra of its sub-Lie algebra of primitive elements. Halperin [11] established the same result working over any subring $R \subset \mathbf{Q}$ containing $1/2$, provided X is a finite CW complex, $H_*(\Omega X; R)$ is torsion-free, and the least non-invertible prime in R is sufficiently large. For a graded R -module M of finite type, set $FM = M/\text{Torsion } M$. Let \mathbf{CW}_q^n be the full subcategory of pointed topological spaces consisting of all finite, q -connected ($q \geq 1$), n -dimensional CW complexes. Suppose $X \in \mathbf{CW}_q^n$, and let P be the sub-Lie algebra of primitive elements of the Hopf algebra $FH_*(\Omega X; R)$. Denote by $\rho(R)$ the least non-invertible prime in R . The aim of this paper is to prove the following generalisation of Halperin's result.

Theorem A. *With the hypotheses and notation above, if $\rho(R) \geq n/q$, then the inclusion $P \subset FH_*(\Omega X; R)$ extends to an isomorphism of Hopf algebras $UP \xrightarrow{\cong} FH_*(\Omega X; R)$.*

Theorem A should not be surprising. Popescu [16] proved it for two-cones X with $\dim X < \rho(R)$. The present author showed that if $X \in \mathbf{CW}_q^n$ and $p \geq n/q$, then $FH_*(\Omega X) \otimes \mathbf{F}_p \cong UL$ for some graded

Date: February 1, 2008.

1991 Mathematics Subject Classification. [.

Key words and phrases. loop space homology, universal enveloping algebra, Bockstein spectral sequence, Hurewicz homomorphism.

This article was written while the author was an NSERC Postdoctoral Fellow.

Lie algebra L [17], and indeed the proof of Theorem A consists of a careful p -local lift of this isomorphism.

Let $X \in \mathbf{CW}_q^n$. Anick has constructed a differential Lie algebra L_X and a chain algebra quasi-isomorphism $\theta_X : UL_X \rightarrow C_*(\Omega X; R)$ that commutes with the respective diagonals up to chain algebra homotopy [2]. It follows that $FH(\theta_X) : FH(UL_X) \rightarrow FH_*(\Omega X; R)$ is an isomorphism of Hopf algebras. Theorem A then follows immediately from the following algebraic result, to be proved in Section 3.

Theorem B. *Let L be a connected, R -free differential Lie algebra of finite type. The inclusion $P \subset FH(UL)$ of the sub-Lie algebra of primitive elements of the Hopf algebra $FH(UL)$ extends to an isomorphism $UP \cong FH(UL)$ of Hopf algebras.*

Our tool for spotting universal enveloping algebras is a differential, torsion-free André-Sjödin theorem. In our context, a differential Γ -Hopf algebra is a differential Hopf algebra which, as an algebra, is an algebra with divided powers.

Theorem C (c.f. [1, 18]). *Let (A, d) be a R -free, commutative differential Hopf algebra of finite type. Then $(A, d)^\sharp \cong U(L, \partial)$ for some differential Lie algebra (L, ∂) if and only if (A, d) is a differential Γ -Hopf algebra.*

The proof of Theorem C is deferred to Section 5, but we will take the result as fact throughout the rest of the paper.

Let $\varphi : \pi_*(\Omega X) \rightarrow H_*(\Omega X)$ be the Hurewicz homomorphism. A theorem of Cartan and Serre (see [9]) asserts that $\varphi \otimes \mathbf{Q}$ is an isomorphism onto the subspace of primitives. Along with the Milnor-Moore theorem, this gives the stunning result that $U(\pi_*(\Omega X) \otimes \mathbf{Q}) \cong H_*(\Omega X; \mathbf{Q})$. The analogous torsion-free result over $R \subset \mathbf{Q}$ does not hold, in general; see Example 6. Instead we have the following theorem.

Theorem D. *With the same hypotheses as in Theorem A, the induced morphism $F\varphi : F\pi_*(\Omega X) \otimes R \rightarrow FH_*(\Omega X; R)$ maps injectively into P , and $P/\text{im } F\varphi$ is torsion.*

A finite, simply-connected complex X is called *elliptic* if $\pi_*(X) \otimes \mathbf{Q}$ is a finite-dimensional vector space: $\sum_{m=1}^{\infty} \dim \pi_m(X) \otimes \mathbf{Q} < \infty$. If X is elliptic, then for sufficiently large primes p , $H_*(\Omega X_{(p)})$ is torsion-free and finitely generated as an algebra [13]. Unfortunately, the notion of “sufficiently large” has not been made precise. As a corollary to Theorems A and D, we obtain the following result along the same lines.

Corollary 1. *If $X \in \mathbf{CW}_q^n$ is elliptic, then $FH_*(\Omega X; R)$ is finitely generated as an algebra, provided $\rho(R) \geq n/q$.*

Proof. For $m \geq 1$, $\dim(\pi_m(\Omega X) \otimes \mathbf{Q}) = \text{rank}(F\pi_m(\Omega X) \otimes R)$. As X is rationally elliptic, the total rank of $F\pi_*(\Omega X) \otimes R$ is finite. By Theorem D, the rank of $P \subset FH_*(\Omega X; R)$ is also finite, and P generates $FH_*(\Omega X; R)$ by Theorem A. \square

The torsion module $P/\text{im } F\varphi$ is an obstruction to the existence of a homotopy product decomposition of ΩX , localised at p , into the product of certain ‘atomic’ spaces. Let \mathcal{C} be the collection of pointed spaces $\{S^{2m-1}, \Omega S^{2m+1}\}_{m \geq 1} \cup \{U^1\} \cup \{T^{2m+1}\{p^r\}\}_{m \geq 2, r \geq 1} \cup \{S^{2m+1}\{p^r\}\}_{m \geq 1, r \geq 1}$. Here $S^{2m+1}\{p^r\}$ is the homotopy-theoretic fibre of the degree p^r self-map on S^{2m+1} , and the spaces $U^1, T^{2m+1}\{p^r\}$ are defined in [8]. Denote by $\prod \mathcal{C}$ the collection of spaces having the weak homotopy type of a finite-type product of spaces in \mathcal{C} .

Theorem 2. *If $X \in \mathbf{CW}_q^n$, $p \geq n/q$, and $\Omega X_{(p)} \in \prod \mathcal{C}$, then $P/\text{im } F\varphi$ vanishes.*

Proof. Let $\varphi^{(r)} : E_\pi^r(\Omega X) \rightarrow E^r(\Omega X)$ be the morphism of Bockstein spectral sequences induced by the Hurewicz morphism [15]. By [3, Theorem 9], if $\Omega X_{(p)} \in \prod \mathcal{C}$, then $E^\infty(\Omega X)$ is generated as an algebra by $\text{im } \varphi^{(\infty)}$. But $E^\infty(\Omega X) \cong FH_*(\Omega X) \otimes \mathbf{F}_p \cong U(P \otimes \mathbf{F}_p)$, so $P \otimes \mathbf{F}_p \subset \varphi^{(\infty)}$. We may identify $\varphi^{(\infty)} = F\varphi \otimes \mathbf{F}_p$, so $p(P/\text{im } F\varphi) = P/\text{im } F\varphi$. The result follows by Nakayama’s Lemma. \square

The author would like to thank Steve Halperin for suggesting the problem, and Ran Levi for many helpful discussions. Don Stanley spotted a small but critical error in an early draft of the paper, and Peter Bubenik’s careful proof-reading helped to make many of the arguments presented here much clearer.

2. REVIEW

2.1. Notation and Conventions. *The ground ring R will always be a principal ideal domain containing $1/2$.* Algebraic objects are graded by the integers and are of finite type. Algebras are augmented to the ground ring; the augmentation ideal is denoted $I(-)$. Topological spaces are pointed and simply-connected.

The dual of a module M is denoted by M^\sharp . Evaluation of $f \in M^\sharp$ at $x \in M$ is denoted $\langle f, x \rangle$. Set $\text{Torsion } M = \{x \in M \mid rx = 0 \text{ for some } r \in R\}$. The *free part* of M is the quotient module $FM = M/\text{Torsion } M$. If $\varphi : M \rightarrow N$ is a linear map, then $\varphi(\text{Torsion } M) \subset \text{Torsion } N$, so φ factors to define a linear map $F\varphi : FM \rightarrow FN$. Let A be a differential Hopf algebra and let $\kappa : H(A) \otimes H(A) \rightarrow H(A \otimes A)$ be the Künneth homomorphism. Then $F\kappa$ is an isomorphism, and so $FH(A)$ is a Hopf algebra.

A *Lie algebra* [7, Section 2] is a non-negatively graded module $L = L_{\geq 0}$ along with a linear morphism $[\cdot, \cdot] : L \otimes L \rightarrow L$ called the Lie

bracket, that satisfies graded anti-symmetry, the graded Jacobi identity, and $[x, [x, x]] = 0$ if $\deg x$ is odd. Since Torsion L is a Lie ideal, FL is a Lie algebra. In fact, if $R = \mathbf{Z}_{(3)}$, and L is a torsion-free module with a bilinear bracket satisfying anti-symmetry and the Jacobi identity, then the last condition is automatically satisfied and so L is a Lie algebra. In particular if X is a space, then $F\pi_*(\Omega X) \otimes \mathbf{Z}_{(3)}$ is a Lie algebra even though $\pi_*(\Omega X) \otimes \mathbf{Z}_{(3)}$ may not be (the canonical example is $\pi_*(\Omega S^{2n})$ [12]). If A is an algebra, then the commutator bracket $[a, a'] = aa' - (-1)^{\deg a \deg a'} a'a$ makes A into a Lie algebra. A Lie algebra is called *connected* if it is concentrated in strictly positive degrees. A *differential Lie algebra* is a chain complex (L, ∂) , where L is a Lie algebra and the differential ∂ satisfies the Leibniz condition $\partial[x, y] = [\partial x, y] + (-1)^{\deg x} [x, \partial y]$. The homology of a differential Lie algebra is a Lie algebra. The *universal enveloping algebra* of a Lie algebra L is an associative algebra UL along with a Lie algebra morphism $\iota : L \rightarrow UL$ such that, if A is an algebra and $f : L \rightarrow A$ is a Lie algebra morphism, then there exists a unique algebra morphism $F : UL \rightarrow A$ such that $F \circ \iota = f$. It follows that the universal enveloping algebra of a Lie algebra is unique up to canonical algebra isomorphism. The Lie algebra morphism $L \rightarrow UL \otimes UL$ defined by $x \mapsto x \otimes 1 + 1 \otimes x$ extends to define a diagonal $\Delta : UL \rightarrow UL \otimes UL$ that provides UL with a natural cocommutative Hopf algebra structure. If L is a differential Lie algebra, then the differential extends to a derivation on UL making it a differential Hopf algebra.

A Γ -algebra is a connected, non-negatively graded commutative algebra $A = A^{\geq 0}$, along with a system of *divided powers* $\gamma^k : A^{2n} \rightarrow A^{2nk}$ for $k \geq 0, n \geq 1$, that satisfy a list of axioms [1]. We will need explicitly the following axiom:

$$\gamma^j(a)\gamma^k(a) = \binom{j+k}{j} \gamma^{j+k}(a); \quad a \in A^{2n}, \quad j, k, n \geq 1.$$

By induction it follows that $a^k = k! \gamma^k(a)$ for $a \in A^{2n}$ and $k, n \geq 1$. Henceforth we assume that all Γ -algebras are augmented. A Γ -morphism is an algebra morphism $\varphi : A \rightarrow B$, where A and B are Γ -algebras and $\varphi(\gamma^k(a)) = \gamma^k(\varphi(a))$ for all $a \in IA^{\text{even}}$ and $k \geq 0$. A Γ -derivation θ on a Γ -algebra A is an algebra derivation that satisfies the additional condition $\theta \gamma^k(a) = \theta(a) \gamma^{k-1}(a)$ for $a \in IA^{\text{even}}$. A *differential Γ -algebra* is a cochain algebra (A, d) , where A is a Γ -algebra and d is a Γ -derivation. A Γ -Hopf algebra is simultaneously a Γ -algebra and a Hopf algebra, such that the diagonal map is a Γ -morphism. A *differential Γ -Hopf algebra* is a differential Hopf algebra (A, d) where A is a Γ -Hopf algebra and d is a Γ -derivation.

2.2. The Bockstein spectral sequence. We review the relevant details of the Bockstein spectral sequence associated to the short exact coefficient sequence $0 \rightarrow \mathbf{Z}_{(p)} \xrightarrow{\times p} \mathbf{Z}_{(p)} \rightarrow \mathbf{F}_p \rightarrow 0$. The standard

reference is Browder [6]; for the particulars of the Bockstein spectral sequence of the universal enveloping algebra of a differential Lie algebra, see [17].

For a $\mathbf{Z}_{(p)}$ -free chain complex C , the Bockstein spectral sequence is a spectral sequence of chain complexes over \mathbf{F}_p . If $H(C)$ is finite type, then the spectral sequence converges, $H(C; \mathbf{F}_p) \Rightarrow FH(C) \otimes \mathbf{F}_p$.

Let (L, ∂) be a connected, $\mathbf{Z}_{(p)}$ -free differential Lie algebra. Then by Theorem C, $(UL)^\sharp$ is a differential Γ -Hopf algebra. For brevity, set $G = (UL)^\sharp$, and let $\{(E_r, \beta_r)\}$ be the Bockstein spectral sequence modulo p for G . For $r \geq 1$, let $\rho_r : H(G) \rightarrow E_r$ be the usual morphism [6, Section 1]. Each ρ_r is an algebra morphism. For each $r \geq 1$ there is an injective differential Γ -morphism $m_{r+1} : (E_{r+1}, 0) \xrightarrow{\sim} (E_r, \beta_r)$ that induces an isomorphism in homology, and is used to identify $E_{r+1} \cong H(E_r)$ as a Γ -Hopf algebra. It follows that for $\zeta \in H(G)$,

$$(1) \quad m_{r+1} \circ \rho_{r+1}(\zeta) = \rho_r(\zeta) + \beta_r(y_r)$$

for some $y_r \in E_r$. For $1 \leq r \leq s$, define injective Γ -morphisms $\sigma_{rs} : E_s \rightarrow E_r$ by $\sigma_{rr} = 1_r : E_r \rightarrow E_r$ for $r \geq 1$ and $\sigma_{rs} = m_s \circ \cdots \circ m_{r+1}$ for $1 \leq r < s$. The sequence $\{E_r\}$ then forms an inverse system, and $E_\infty = \varprojlim E_r$. Using this characterisation, one can show that E_∞ is a Γ -Hopf algebra; in particular, by [18], $E_\infty \cong \Gamma(V_\infty)$ as a Γ -algebra. The map $\sigma_\infty = \varprojlim \sigma_{1r}$ is an injective Γ -morphism. Finally, $\rho_\infty = \varprojlim \rho_r : H(G) \rightarrow E_\infty$ is a surjective algebra morphism, and $\ker \rho = \text{Torsion}(H(G)) + pH(G)$.

3. HOMOLOGY OF UL MODULO TORSION

In this section we prove the main algebraic result of the paper, Theorem B from the introduction, and discuss some properties of the morphism $FH(L) \rightarrow FH(UL)$ induced by the canonical inclusion $L \rightarrow UL$. Since UL is a differential Hopf algebra, $FH(UL)$ is a Hopf algebra.

Proof of Theorem B. Let $P \subset FH(UL)$ be the submodule of primitive elements. The inclusion extends to a Hopf algebra morphism $\beta : UP \rightarrow FH(UL)$. Recall that β is an isomorphism if and only if $\beta_{\mathfrak{p}} : (UP)_{\mathfrak{p}} \rightarrow FH(UL)_{\mathfrak{p}}$ is an isomorphism for all prime ideals \mathfrak{p} in R [4]. Furthermore, each prime ideal \mathfrak{p} in R is of the form pR for some prime $p \geq \rho(R)$, and $R_{\mathfrak{p}} \cong \mathbf{Z}_{(p)}$. Therefore, without loss of generality, we may assume that L is a differential Lie algebra over $\mathbf{Z}_{(p)}$, $p \geq \rho(R)$.

To begin, we construct a subalgebra A of $H[(UL)^\sharp]$ such that $A \otimes \mathbf{F}_p \cong E_\infty$. Recall (Section 2.2) that $E_\infty \cong \Gamma(V_\infty)$. Let $\{v_i\}$ be a basis of V_∞ . Since ρ_∞ is surjective, each $v_i = \rho_\infty(\zeta_i)$ for some $\zeta_i \in H(G)$. Let $n = \deg v_i$. There exists $N > 0$ such that $\beta_r(E_r^{\leq n}) = 0$ whenever $r \geq N$. It follows from (1) that $\sigma_\infty(v_i) = \rho_1(\zeta_i + \sum_{r=1}^N \xi_r)$ where $\xi_r \in H(G)$ has order p^r . Let $z_i \in G$ be a cycle whose homology class is $\zeta_i + \sum_{r=1}^N \xi_r$. Let V be the free $\mathbf{Z}_{(p)}$ -module on the basis $\{v_i\}$. Define a

linear map $\sigma : V \rightarrow G$ by $\sigma(v_i) = z_i$. Then σ extends to a differential Γ -morphism $A = (\Gamma(V), 0) \xrightarrow{\sigma} G$, and $\sigma \otimes \mathbf{F}_p = \sigma_\infty$. Since σ_∞ is injective and $\Gamma(V)$, G are $\mathbf{Z}_{(p)}$ -free as modules, σ is injective.

Let $\pi : A \rightarrow E_\infty$ be reduction mod p . Passing to homology, it is a routine exercise to verify that $\rho_1 \circ \sigma_* = \sigma_\infty \circ \pi$ and $\rho_\infty \circ \sigma_* = \pi$. Suppose $\sigma_*(a) = 0$ for some $a \in A$. Then $\pi(a) = \rho_\infty \circ \sigma_*(a) = 0$, so $a = pa'$ for some $a' \in A$. Thus $p\sigma_*(a') = 0$, so $\sigma_*(a')$ is a torsion element of $H(G)$. It follows that $\rho_\infty \circ \sigma_*(a') = 0$, whence $\pi(a') = 0$. So $a' = pa''$ for some $a'' \in A$, which is again sent to a torsion element of $H(G)$ by σ_* . Repeat the argument; eventually the process ends since A has no infinitely p -divisible elements. Therefore $a = 0$ and so σ_* is injective.

Let T be the torsion ideal of $H(G)$. Since $T \subset \ker \rho_\infty$, the morphism ρ_∞ factors to define $\bar{\rho}_\infty : FH(G) \rightarrow E_\infty$. Let $\hat{\sigma}_*$ be the composite $A \xrightarrow{\sigma_*} H(G) \rightarrow FH(G)$. One verifies that $\bar{\rho}_\infty \circ \hat{\sigma}_* = \pi$. From the definitions, $\hat{\sigma}_* \otimes \mathbf{F}_p$ is the identity on E_∞ . Thus $FH(G)/\text{im } \hat{\sigma}_* = p(FH(G)/\text{im } \hat{\sigma}_*)$. Since $FH(G)/\text{im } \hat{\sigma}_*$ is degree-wise finitely generated, Nakayama's Lemma states that $FH(G)/\text{im } \hat{\sigma}_* = 0$. Therefore $\hat{\sigma}_*$ is surjective and hence an isomorphism of algebras.

By Theorem C, G is a differential Γ -Hopf algebra. We claim that the diagonal ψ in G induces a Γ -Hopf algebra structure on A . Indeed, since σ is a Γ -morphism, $\sigma_*(\gamma^k(a)) = [\gamma^k(\sigma(a))]$. A Γ -morphism $\chi : A \rightarrow A \otimes A$ is defined by $(\hat{\sigma}_*^{-1} \otimes \hat{\sigma}_*^{-1}) \circ FH(\psi) \circ \hat{\sigma}_*$. Since ψ is a Γ -morphism, so too is χ . Furthermore, χ inherits coassociativity and the counit from ψ , and so χ provides A with a Γ -Hopf algebra structure.

By definition, χ commutes with σ “up to torsion”. That is, given $a \in A$, there exist $\Phi, \Psi \in G$, and $r \geq 0$ such that

$$(2) \quad (\sigma \otimes \sigma) \circ \chi(a) = \psi \circ \sigma(a) + \Phi$$

and $d\Psi = p^r \Phi$.

We now define a pairing

$$(3) \quad A \otimes FH(UL) \xrightarrow{\langle \cdot, \cdot \rangle} \mathbf{Z}_{(p)}$$

that exhibits $FH(UL)$ as the Hopf algebra dual of the Γ -Hopf algebra A . By Theorem C, the proof will then be complete. Use σ to identify $(A, 0)$ as a differential Γ -subalgebra of G to obtain a pairing $A \otimes \ker \partial \xrightarrow{\langle \cdot, \cdot \rangle} \mathbf{Z}_{(p)}$, where ∂ is the differential on UL . Let $T = \{x \in UL \mid p^r x \in \text{im } \partial \text{ for some } r \geq 0\}$. Since the differential in A vanishes, $\langle A, \text{im } \partial \rangle = 0$. Furthermore, $\mathbf{Z}_{(p)}$ is a domain, hence $\langle A, T \rangle = 0$. Therefore $\langle \cdot, \cdot \rangle$ factors to define (3).

Observe that (3) is non-singular. Indeed, as a chain map, $\sigma : (A, 0) \rightarrow G$ splits, and σ induces an isomorphism $A \cong FH(G)$. It follows that $G = A \oplus B \oplus C$, where $d : B \rightarrow C$ and $H(G) = A \oplus (C/dB)$. Dually, $UL = A^\# \oplus B^\# \oplus C^\#$, and $H(UL) = A^\# \oplus (B^\#/\partial C^\#)$, where $B^\#/\partial C^\#$ is the torsion submodule. Therefore $FH(UL) \cong A^\#$ as a module, and one can check that the duality is provided by (3).

Since σ is an algebra morphism, $\langle a \otimes a', \Delta x \rangle = \langle aa', x \rangle$ for $x \in UL$ and $a, a' \in A$. Pass to (3) to see that the multiplication in A is dual to the diagonal $FH(\Delta)$.

Suppose that $x, y \in UL$ are cycles, and $a \in A$. Using (2), we see that $\langle a, xy \rangle = \langle \psi(a), x \otimes y \rangle = \langle \chi(a), x \otimes y \rangle + \langle \Phi, x \otimes y \rangle$, where $p^r \Phi = d\Psi$ for some $\Psi \in UL \otimes UL$. Since $p^r \Phi$ is a boundary, x and y are cycles, and $\mathbf{Z}_{(p)}$ is an integral domain, it follows that $\Phi(x \otimes y) = 0$. Pass to (3) to conclude that χ is dual to the multiplication in $FH(UL)$. \square

Proposition 3. *The natural map $\iota : L \rightarrow UL$ induces an injection of Lie algebras $F\iota_* : FH(L) \hookrightarrow P$, with $P/\text{im } F\iota_*$ torsion.*

Proof. Let $\bar{\Delta}$ be the reduced diagonal on UL , and let $\kappa : H(UL) \otimes H(UL) \rightarrow H(UL \otimes UL)$ be the Künneth morphism. Recall that $F\kappa$ is an isomorphism. Since $\bar{\Delta} \circ \iota = 0$, $(F\kappa)^{-1} \circ F\bar{\Delta}_* \circ F\iota_* = 0$, and so $\text{im } F\iota_* \subset P$. Observe that $FH(L) \otimes \mathbf{Q} = H(L \otimes \mathbf{Q})$ and $FH(UL) \otimes \mathbf{Q} = H(U(L \otimes \mathbf{Q})) = UH(L \otimes \mathbf{Q})$. Furthermore, $F\iota_* \otimes \mathbf{Q}$ is the canonical inclusion $H(L \otimes \mathbf{Q}) \rightarrow UH(L \otimes \mathbf{Q})$. As $FH(L)$ and $FH(UL)$ are torsion-free, the result now follows. \square

Example 4. Define a differential Lie algebra L as follows. As a chain complex, L is the free $\mathbf{Z}_{(3)}$ -module on the graded basis $\{x_1, y_2, x_3, x_5\}$, with subscript indicating degree; the only non-zero differential is $dy_2 = 3x_1$. The bracket is defined by $[y_2, x_1] = x_3$, $[y_2, x_3] = x_5$, and all others vanishing. Let $\langle S \rangle$ denote the abelian Lie algebra on the graded set S . Then $FH(L) = \langle [x_3], [x_5] \rangle$.

By computing the Bockstein spectral sequence for UL , one finds that $FH(UL) = U\langle z_3, z_5 \rangle$, where z_3 and z_5 are represented in UL by x_3 and $x_1y_2^2 + y_2x_3$, respectively. Let $\iota : L \rightarrow UL$ be the canonical injection. Clearly $F\iota_*[x_3] = z_3$; however, $dy_2^3 = -6x_5 + 9(x_1y_2^2 + y_2x_3)$ in UL , and so $F\iota_*[x_5] = (3/2)z_5$. Therefore, $P/\text{im } F\iota_* \cong \mathbf{Z}/3\mathbf{Z}$, with generator represented by z_5 .

4. THE HUREWICZ HOMOMORPHISM

In this section we prove Theorem D. Suppose $X \in \mathbf{CW}_q^n$. As usual, set $P = P(FH_*(\Omega X; R))$.

We will need the following lemma. Let $Q \subset H_*(\Omega X; \mathbf{Q})$ be the sub Lie algebra of primitive elements. The inclusion extends to an isomorphism of Hopf algebras $\beta : UQ \cong H_*(\Omega X; \mathbf{Q})$ [14].

Lemma 5. $P \otimes \mathbf{Q} \cong Q$.

Proof. By universal coefficients, the natural Hopf algebra morphism $\mu : H_*(\Omega X; R) \otimes \mathbf{Q} \rightarrow H_*(\Omega X; \mathbf{Q})$, defined by $\mu([z] \otimes x) = [z \otimes x]$ for $x \in \mathbf{Q}$ and cycles $z \in C_*(\Omega X; R)$, is an isomorphism. Let $r : H_*(\Omega X; R) \rightarrow FH_*(\Omega X; R)$ be the quotient morphism. Since $r \otimes \mathbf{Q} : H_*(\Omega X; R) \otimes \mathbf{Q} \rightarrow [FH_*(\Omega X; R)] \otimes \mathbf{Q}$ is an isomorphism, we get an

isomorphism of Hopf algebras $\bar{\mu} : [FH_*(\Omega X; R)] \otimes \mathbf{Q} \xrightarrow{\cong} H_*(\Omega X; \mathbf{Q})$ defined by $\bar{\mu} = \mu \circ (r \otimes \mathbf{Q})^{-1}$. By Theorem A and the mapping property of universal enveloping algebras, $\alpha \otimes \mathbf{Q} : U(P \otimes \mathbf{Q}) \xrightarrow{\cong} [FH_*(\Omega X; R)] \otimes \mathbf{Q}$. It follows that $\beta^{-1} \circ \bar{\mu} \circ (\alpha \otimes \mathbf{Q})$ restricts to an isomorphism of Lie algebras $P \otimes \mathbf{Q} \xrightarrow{\cong} Q$. \square

Proof of Theorem D. Let $\varphi : \pi_*(\Omega X) \rightarrow H_*(\Omega X)$ be the Hurewicz homomorphism. The inclusion $R \subset \mathbf{Q}$ induces the commutative diagram

$$\begin{array}{ccc} \pi_*(X) \otimes R & \xrightarrow{\varphi \otimes R} & H_*(\Omega X; R) \\ \downarrow & & \downarrow \\ \pi_*(\Omega X) \otimes \mathbf{Q} & \xrightarrow{\varphi \otimes \mathbf{Q}} & H_*(\Omega X; \mathbf{Q}) \end{array}$$

in which the right arrow is a Hopf algebra morphism, and the other arrows preserve brackets. The kernels of the vertical arrows are precisely the respective torsion submodules. Therefore the diagram

$$\begin{array}{ccccc} F\pi_*(X) \otimes R & \xrightarrow{F\varphi \otimes R} & FH_*(\Omega X; R) & \xleftarrow[\alpha]{\cong} & UP \\ \downarrow & & \downarrow & & \downarrow \\ \pi_*(\Omega X) \otimes \mathbf{Q} & \xrightarrow{\varphi \otimes \mathbf{Q}} & H_*(\Omega X; \mathbf{Q}) & \xleftarrow[\beta]{\cong} & U(P \otimes \mathbf{Q}) \end{array}$$

commutes. The left vertical arrow, $F\varphi \otimes R$, and $\varphi \otimes \mathbf{Q}$ are Lie algebra morphisms. The remaining arrows are Hopf algebra morphisms. The vertical arrows are injections, $\varphi \otimes \mathbf{Q}$ is an isomorphism onto $P \otimes \mathbf{Q}$ by Lemma 5, and the right vertical arrow is of the form $U\{P \rightarrow P \otimes \mathbf{Q}\}$. It follows that $F\varphi \otimes R : F\pi_*(X) \otimes R \rightarrow P$ is an injection and that $P/\text{im } F\varphi \otimes R$ is torsion. \square

Example 6. For this example, all spaces are localised at a fixed prime p . Let $\alpha : S^{2p} \rightarrow S^3$ be a representative of the generator of $\pi_{2p}(S^3) \cong \mathbf{Z}/p\mathbf{Z}$. Set $K = S^4 \cup_{\Sigma\alpha} e^{2p+2}$. The map $\Sigma\alpha \circ p : S^{2p+1} \rightarrow S^4$ is null-homotopic, and so we get a map $f : S^{2p+2} \rightarrow K$ making the diagram

$$\begin{array}{ccccc} S^{2p+1} & \longrightarrow & * & \longrightarrow & S^{2p+2} \\ p \downarrow & & \downarrow & & \downarrow f \\ S^{2p+1} & \xrightarrow{\Sigma\alpha} & S^4 & \longrightarrow & K \end{array}$$

commute up to homotopy, where the rows are cofibration sequences. From the associated long exact sequence in homology, we deduce that in degree $2p+2$, f_* is multiplication by p .

By the Bott-Samelson theorem, $H_*(\Omega K) \cong T(e_3, e_{2p+1})$ and $P = PH_*(\Omega K) = L(e_3, e_{2p+1})$. Let $\varphi : \pi_*(\Omega K) \rightarrow H_*(\Omega K)$ be the Hurewicz map. Then $\varphi(f^\sharp) = pe_{2p+1}$, where $f^\sharp : S^{2p+1} \rightarrow \Omega K$ is the adjoint

of f . Since $\varphi(f^\sharp) \neq 0$ and $H_*(\Omega K)$ is torsion-free, f^\sharp is not a torsion element. It follows that $pe_{2p+1} \in \text{im } F\varphi$.

If $e_{2p+1} \in \text{im } F\varphi$, then $e_{2p+1} \in \text{im } \varphi$. By Baues [5, Lemma V.3.10], ΩK would be homotopy-equivalent to the product of odd-dimensional spheres and loops on the same. But K is a retract of $\Sigma^2 \mathbf{C}P^p$, and so $H^*(\Omega K; \mathbf{F}_p)$ has a non-vanishing \mathcal{P}^1 . Therefore $e_{2p+1} \notin \text{im } F\varphi$, and so $P/\text{im } F\varphi \cong \mathbf{Z}/p\mathbf{Z}$.

5. A TORSION-FREE ANDRÉ-SJÖDIN THEOREM

This section is devoted to the proof of Theorem C. Let A be an R -free Γ -algebra. The module $\text{Hom}(A, A)$ is an algebra under composition, and hence is a Lie algebra with the commutator bracket. The submodule $\text{Der } A$ of all Γ -derivations on A is a sub-Lie algebra of $\text{Hom}(A, A)$.

We will need a construction of Gulliksen and Levin [10]. Suppose A is an R -free Γ -Hopf algebra, with diagonal Δ . Given $f \in A^\sharp$, define $\nu(f)$ to be the composite

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{1 \otimes f} A \otimes R = A.$$

Define $DA = IA \cdot IA + \sum_{k \geq 2} \gamma^k(IA^{\text{even}})$. Set $\tilde{P}(A^\sharp) = \{f \in A^\sharp \mid f(DA) = 0\}$. The primitives of A^\sharp may be identified as $P(A^\sharp) = \{f \in A^\sharp \mid f(IA \cdot IA) = 0\}$ [14].

Lemma 7. [18] *With the notation and hypotheses above,*

- (a) $\nu : A^\sharp \rightarrow \text{Hom}(A, A)$ is an algebra morphism;
- (b) $\nu(f)|_{A^{\deg f}} = f$;
- (c) $\nu(f) \in \text{Der } A$ if and only if $f \in \tilde{P}(A^\sharp)$;
- (d) $\tilde{P}(A^\sharp)$ is a sub-Lie algebra of A^\sharp and $\nu : \tilde{P}(A^\sharp) \rightarrow \text{Der } A$ is a Lie monomorphism.

Proposition 8. *Let A be a $\mathbf{Z}_{(p)}$ -free Γ -Hopf algebra over $\mathbf{Z}_{(p)}$.*

- (a) *There is a submodule $X \subset A$ whose inclusion extends to an isomorphism of Γ -algebras $\Gamma(X) \xrightarrow{\cong} A$.*
- (b) *The inclusion $P(A^\sharp) \subset A^\sharp$ extends to an isomorphism of Hopf algebras $UP(A^\sharp) \xrightarrow{\cong} A^\sharp$.*

Proof. (a) Set $\bar{A} = A \otimes \mathbf{F}_p$. Then \bar{A} is a Γ -Hopf algebra over \mathbf{F}_p . Let $\sigma : \bar{A}/D\bar{A} \rightarrow \bar{A}$ be a splitting of the quotient map. By Sjödin [18], σ extends to an isomorphism of Γ -algebras $\Gamma(\bar{A}/D\bar{A}) \xrightarrow{\cong} \bar{A}$. Let $\{\bar{x}_i\}$ be a basis of $\text{im } \sigma \subset \bar{A}$ indexed over the positive integers. Choose a set of representatives $\{x_i\}$ in A , and let X be the linear span. The inclusion $X \subset A$ extends to a Γ -morphism $\varphi : \Gamma(X) \rightarrow A$. The reduction mod p , $\varphi \otimes \mathbf{F}_p : \Gamma(\text{im } \sigma) \rightarrow \bar{A}$, is an isomorphism. In particular, $\varphi \otimes \mathbf{F}_p$ is surjective, so by Nakayama's Lemma, $A/\text{im } \varphi = 0$. It follows that φ is surjective. Since A and $\Gamma(X)$ are $\mathbf{Z}_{(p)}$ -free and finite type, and $\varphi \otimes \mathbf{F}_p$ is injective, φ is itself injective and therefore an isomorphism.

(b) To begin, we claim that $P(A^\sharp) = \tilde{P}(A^\sharp)$. From the definitions, $\tilde{P}(A^\sharp) \subset P(A^\sharp)$. It suffices to show that if $f \in P(A^\sharp)$, then $f(\gamma^k(a)) = 0$ for all $k \geq 2$ and $a \in IA^{\text{even}}$. Since $a^k = k!\gamma^k(a)$ and $f(IA \cdot IA) = 0$, $0 = f(a^k) = k!f(\gamma^k(a))$. As $\mathbf{Z}_{(p)}$ is a domain, the claim follows.

To show that the canonical algebra morphism $UP(A^\sharp) \rightarrow A^\sharp$ is surjective, we use an argument of Sjödín. As a module, $IA \cong X \oplus DA$. For $i \geq 1$, define $f_i \in A^\sharp$ by $f_i(\mathbf{Z}_{(p)} \oplus DA) = 0$ and $f_i(x_j) = \delta_{ij}$. By definition, each $f_i \in P(A^\sharp)$. Let \mathcal{A} be the set of finite sequences of non-negative integers $r = (r_i)$ such that $r_i = 0$ or 1 if $\deg x_i$ is odd, and the last term in r is non-zero. Define $n(r)$ to be the length of r (so $r_{n(r)} \neq 0$). Order \mathcal{A} by setting $r > r'$ if $n(r) > n(r')$ or $n(r) = n(r')$ and $r_{n(r)} > r'_{n(r')}$. Set $f^r = f_1^{r_1} \cdots f_n^{r_n}$ and $\gamma^r(x) = \gamma^{r_n}(x_n) \cdots \gamma^{r_1}(x_1)$ where $n = n(r)$. Since ν is an algebra morphism, $\nu(f^r) = \nu(f_1)^{r_1} \circ \cdots \circ \nu(f_n)^{r_n}$ (exponents with respect to composition). As $f_i \in P(A^\sharp) = \tilde{P}(A^\sharp)$, $\nu(f_i)$ is a Γ -derivation; that is, $\nu(f_i)(\gamma^k(x_i)) = \nu(f_i)(x_i)\gamma^{k-1}(x_i)$. It follows that $\nu(f^r)(\gamma^r(x)) = 1$, and $\nu(f^r)(\gamma^{r'}(x)) = 0$ if $r > r'$. By Lemma 7(b), $f^r(\gamma^r(x)) = 1$, while $f^r(\gamma^{r'}(x)) = 0$ if $r > r'$ and $\deg f^r = \deg \gamma^{r'}(x)$. If $r > r'$ and $\deg f^r \neq \deg \gamma^{r'}(x)$, then of course $f^r(\gamma^{r'}(x)) = 0$ for degree reasons. The set $\{\gamma^r(x)\}_{r \in \mathcal{A}}$ forms an additive basis for A . The above formulae for the f^r imply that $\{f^r\}_{r \in \mathcal{A}}$ is a basis for A^\sharp . In particular, the set $\{f_i\}$ generates A^\sharp as an algebra, and so $UP(A^\sharp) \rightarrow A^\sharp$ is surjective.

Let χ_V denote the Euler-Poincaré series of the non-negatively graded \mathbf{F}_p -space V . We extend the definition to a non-negatively graded, free $\mathbf{Z}_{(p)}$ -module M : $\chi_M = \sum_{n \geq 0} (\text{rank } M_n) t^n$. Note that $\chi_M = \chi_{M \otimes \mathbf{F}_p}$. Since $UP(A^\sharp) \rightarrow A^\sharp$ is surjective, $\chi_{UP(A^\sharp)} \geq \chi_{A^\sharp}$. If $f \in P(A^\sharp)$, then $(f \otimes \mathbf{F}_p)(D\bar{A}) = 0$, so $P(A^\sharp) \otimes \mathbf{F}_p \subset \tilde{P}(\bar{A}^\sharp)$ and the natural morphism $U(P(A^\sharp) \otimes \mathbf{F}_p) \rightarrow U\tilde{P}(\bar{A}^\sharp) = \bar{A}^\sharp$ is an injection. It follows that $\chi_{UP(A^\sharp)} \leq \chi_{A^\sharp}$, so in fact the Euler-Poincaré series are equal. Therefore $UP(A^\sharp) \rightarrow A^\sharp$ is an isomorphism. \square

Before finally proving Theorem C, we state a Lemma.

Lemma 9. *Let R be a characteristic-zero principal ideal domain. Let A and B be R -free Γ -algebras.*

- (1) *If $\varphi : A \rightarrow B$ is an algebra morphism, then it is a Γ -morphism.*
- (2) *If $\theta : A \rightarrow B$ is a derivation, then it is a Γ -derivation.*

In particular, if A is a Γ -algebra and a differential Hopf algebra, then A is a differential Γ -Hopf algebra.

Proof. Straightforward. \square

Proof of Theorem C. Suppose A is a differential Γ -Hopf algebra. As usual, denote by $P(A^\sharp)$ the sub-Lie algebra of primitive elements. The natural Hopf algebra morphism $UP(A^\sharp) \rightarrow A^\sharp$ is an isomorphism if and only if $UP(A^\sharp)_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}^\sharp$ is an isomorphism for every prime ideal \mathfrak{p} of R , and that each prime ideal \mathfrak{p} is of the form pR for some prime

$p \geq \rho(R)$. Furthermore, observe that $P(A^\#)_{(p)} = P(A^\#_{(p)})$. Therefore, without loss of generality we may assume that $R = \mathbf{Z}_{(p)}$ for some odd prime p . By Proposition 8, $A^\# \cong UP(A^\#)$ as a Hopf algebra. Since the differential in $A^\#$ is a Hopf derivation, it preserves $P(A^\#)$. Thus $P(A^\#)$ forms a differential Lie algebra, establishing the ‘if’ assertion.

Conversely, suppose $(A, d)^\# \cong U(L, \partial)$. By Lemma 9, it suffices to show that $A^\#$ is a Γ -algebra. In [11], Halperin constructed a chain isomorphism $\gamma_L : UL \rightarrow (\Gamma(L^\#), \bar{D})^\#$, where \bar{D} is a Γ -derivation. By construction, γ_L factors as

$$UL \xrightarrow{U\sigma} U\mathcal{D} \xrightarrow{\theta} (\Gamma(L^\#), \bar{D})^\#$$

for some differential Lie morphism $\sigma : L \rightarrow \mathcal{D}$. By the proof of [11, Proposition 4.2], θ is a coalgebra morphism. Since $U\sigma$ is also a coalgebra morphism, γ_L is an isomorphism of chain coalgebras. Dualising provides $A \cong (UL)^\#$ with the required Γ -algebra structure. \square

REFERENCES

1. M. André, *Hopf algebras with divided powers*, J. Algebra **18** (1971), 19–50.
2. David J. Anick, *Hopf algebras up to homotopy*, J. Amer. Math. Soc. **2** (1989), no. 3, 417–453.
3. ———, *Single loop space decompositions*, Trans. Amer. Math. Soc. **334** (1992), no. 2, 929–940.
4. M.F. Atiyah and I.G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, London, 1969.
5. Hans Joachim Baues, *Commutator calculus and groups of homotopy classes*, London Mathematical Society Lecture Note Series, no. 50, Cambridge University Press, Cambridge, 1981.
6. William Browder, *Torsion in H-spaces*, Ann. of Math. (2) **74** (1961), 24–51.
7. F.R. Cohen, J.C. Moore, and J.A. Neisendorfer, *Torsion in homotopy groups*, Ann. of Math. (2) **109** (1979), 121–168.
8. ———, *Exponents in homotopy theory*, Algebraic Topology and Algebraic K-Theory (Princeton, New Jersey) (William Browder, ed.), Annals of Mathematics Studies, no. 113, Princeton University Press, 1987, pp. 3–34.
9. Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, Berlin/Heidelberg, 2001.
10. Tor H. Gulliksen and Gerson Levin, *Homology of local rings*, Queen’s Papers in Pure and Applied Mathematics, no. 20, Queen’s University, Kingston, Ontario, 1969.
11. Stephen Halperin, *Universal enveloping algebras and loop space homology*, J. Pure Appl. Alg. **83** (1992), 237–282.
12. Arunas Liulevicius, *The Factorization of cyclic reduced powers by secondary cohomology operations*, Mem. Amer. Math. Soc. (1962), no. 42.
13. C. A. McGibbon and C. W. Wilkerson, *Loop spaces of finite complexes at large primes*, Proc. Amer. Math. Soc. **96** (1986), no. 4, 698–702.
14. John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.
15. Joseph Neisendorfer, *Primary homotopy theory*, Mem. Amer. Math. Soc. **25** (1980), no. 232.
16. Călin Popescu, *Characteristic zero loop space homology of two-cones*, Bull. London Math. Soc. **32** (2000), 600–608.

17. Jonathan A. Scott, *Algebraic structure in loop space homology*, Ph.D. thesis, University of Toronto, Toronto, Canada, 2000.
18. Gunnar Sjödin, *Hopf algebras and derivations*, J. Algebra **64** (1980), 218–229.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ABERDEEN, ABERDEEN AB24 3UE, UNITED KINGDOM

E-mail address: `j.scott@maths.abdn.ac.uk`